LYAPUNOV FUNCTIONALS FOR FOURTH-ORDER LUBRICATION EQUATIONS

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Abstract: An algorithmic approach, based on systematic treatment of integration by parts, has been used to obtain a class of Lyapunov functionals and corresponding dissipation bounds for several fourth-order lubrication equations.

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1 INTRODUCTION

Evolution equations of fourth order in spatial derivatives emerge in large variety of physical systems [4, 9, 10]. Besides the historically impotant Cahn-Hilliard equation, which describes a phase separation process in material science [9], the most prominent fourth-order model is the thin-film equation

$$u_t + (u^\beta u_{xxx})_x = 0. \tag{TF}$$

This equation arises in lubrication approximation theory of thin viscous fluids [4], with $u(t,x) \geq 0$ describing the fluid height. Physically relevant values of the parameter β are $\beta = 1$ [8], $\beta = 2$ and $\beta = 3$ [4, 12], but mathematical curiosity reveals that different values of β yield striking differences in properties of solutions. For instance, if $\beta \geq 3.5$, interior finite time singularities are not possible [5], while source type solutions with compact support exist only if $0 < \beta < 3$ [2].

In [3] Bertozzi lists other three lubrication equations, all having the same fourth-order term:

$$u_t + u^\beta u_{xxxx} = 0, \tag{MTF}$$

$$u_t + (u^\beta u_{xx})_{xx} = 0, \tag{SATF}$$

$$u_t + (u^\beta u_x)_{xxx} = 0. \tag{ATF}$$

The first one is the modified thin-film equation (MTF), which compared to (TF) lacks the convective term $\beta u^{\beta-1} u_{xxx} u_x$. The last two equations are mathematically interesting: (SATF) having the self-adjoint structure of the nonlinearity and (ATF) being formal adjoint to (TF). In the rest of the paper we assume that initial conditions (at $t = 0$) to the above equations are given by a nonnegative function u_0 and periodic boundary conditions are imposed, i.e. we consider domain being the onedimensional torus $\mathbb{T} \simeq [0,1)$.

Many nonlinear equations share a rich dissipative structure — the Lyapunov property of certain nonlinear functionals, called entropies, along respective solution trajectories. This also gives rise to desired a priori estimates, which are crucial in the existence analysis, as well as in the treatment of other qualitative properties of solutions. Such estimates are particularly important in the analysis of higher-order equations, since they do not obey the maximum principle.

The thin-film equation (TF) is relatively well studied model, see [1], but the other lubrication equations did not receive much attention in the literature. In this paper we study dissipation properties of positive classical solutions to equations (MTF), (SATF) and (ATF). We provide a class of functionals E_α (see Definition 1 below) that are nonincreasing in time (Lyapunov) along respective solutions. Moreover, we obtain generic bounds on the rate of dissipation, so called entropy production inequalities. Such properties are known for the thin-film equation: E_α are Lyapunov functionals provided $3/2 \leq \alpha + \beta \leq 3$, and stronger dissipation bounds hold for $3/2 < \alpha + \beta < 3$ [6, 11]. Concerning the other three lubrication equations, only entropy conservation property for equation (MTF) is known for $\alpha + \beta = 1$ AND $\beta \neq 0$ [3]. Our results on dissipation structure of these lubrication equations, presented in Theorem 1 and 2 below, are to the best of our knowledge new in the literature.

In order to accomplish the work, we use an algorithmic approach presented by Jüngel and Matthes [11], where integration by parts formulae — the key tool in deriving integral bounds — are treated in a systematic way. That is the topic of the next section.

2 Dissipation properties of lubrication equations

Let us start with our basic definitions.

Definition 1 *Let u be a nonnegative solution to any of the above lubrication equations, then*

$$E_\alpha[u(t)] = \frac{1}{\alpha(\alpha-1)} \int_{\mathbb{T}} u^\alpha \mathrm{d}x \quad \textit{for all } t \geq 0, \quad \alpha \neq 0, 1, \tag{1}$$

defines an α-functional called entropy *or* Lyapunov functional *if there exists a constant $c \geq 0$ and a well-defined nonnegative functional P_α such that the following* entropy production inequality *holds*

$$\frac{\mathrm{d}}{\mathrm{d}t} E_\alpha[u(t)] + cP_\alpha[u(t)] \leq 0 \quad \textit{for all } t \geq 0. \tag{2}$$

Note 1 *For $\alpha = 0$ and 1, the limit cases, the α-functionals are defined by $E_0[u] = \int_{\mathbb{T}}(u - \log u)\mathrm{d}x$ and $E_1[u] = \int_{\mathbb{T}}(u(\log u - 1) + 1)\mathrm{d}x$, which is (up to the sign) the Boltzmann entropy. Functional P_α is called entropy production and it is typically defined in terms of u and its spatial derivatives, see* (7) *below.*

2.1 The modified thin-film equation

Let $u : (0, \infty) \times \mathbb{T} \to \mathbb{R}^+$ be a smooth and strictly positive solution to (MTF). We calculate

$$-\frac{\mathrm{d}}{\mathrm{d}t} E_\alpha[u] = -\frac{1}{\alpha - 1} \int_{\mathbb{T}} u^{\alpha-1} u_t \mathrm{d}x = \frac{1}{\alpha-1} \int_{\mathbb{T}} u^{\alpha+\beta} \Big(\frac{u_{xxxx}}{u}\Big) \mathrm{d}x. \tag{3}$$

In order to prove the nonnegativity of the last integral, we use *all possible* integration by parts formulae. It has been proved in [11], that — based on periodic boundary conditions and particular structure of the integrand — all integration by parts formulae share the following algebraic representation

$$I = \int_{\mathbb{T}} \Big(u^{\alpha+\beta} R\Big(\frac{u_x}{u}, \frac{u_{xx}}{u}, \frac{u_{xxx}}{u}\Big)\Big)_x \mathrm{d}x = \int_{\mathbb{T}} u^{\alpha+\beta} T\Big(\frac{u_x}{u}, \frac{u_{xx}}{u}, \frac{u_{xxx}}{u}, \frac{u_{xxxx}}{u}\Big) = 0 \tag{4}$$

with $R(\xi_1, \xi_2, \xi_3)$ and $T(\xi_1, \xi_2, \xi_3, \xi_4)$ being polynomials in real variables, which are linear combinations of monomials $\xi_1^{p_1}\xi_2^{p_2}\xi_3^{p_3}$ and $\xi_1^{p_1}\xi_2^{p_2}\xi_3^{p_3}\xi_4^{p_4}$ that satisfy $p_1 + 2p_2 + 3p_3 = 3$ and $p_1 + 2p_2 + 3p_3 + 4p_4 = 4$, respectively. Based on that observation, there are only three basic integration by parts rules, namely those corresponding to the nonnegative integer solutions of $p_1 + 2p_2 + 3p_3 = 3$. These are $R_1(\xi_1, \xi_2, \xi_3) = \xi_1^3$, $R_2(\xi_1, \xi_2, \xi_3) = \xi_1\xi_2$ and $R_3(\xi_1, \xi_2, \xi_3) = \xi_3$ with it's corresponding *shift polynomials*. Respectively: $T_1(\xi) = (\alpha + \beta - 3)\xi_1^4 + 3\xi_1^2\xi_2$, $T_2(\xi) = (\alpha + \beta - 2)\xi_1^2\xi_2 + \xi_2^2 + \xi_1\xi_3$ and $T_3(\xi) = (\alpha + \beta - 1)\xi_1\xi_3 + \xi_4$.

Since $I_1 = I_2 = I_3 = 0$, we can write for arbitrary real coefficients c_1, c_2 and c_3,

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}t} E_\alpha[u] &= \int_{\mathbb{T}} u^{\alpha+\beta} S_0\Big(\frac{u_x}{u}, \ldots, \frac{u_{xxxx}}{u}\Big)\mathrm{d}x + c_1 I_1 + c_2 I_2 + c_3 I_3 \\ &= \int_{\mathbb{T}} u^{\alpha+\beta} \big(S_0 + c_1T_1 + c_2T_2 + c_3T_3\big)\Big(\frac{u_x}{u}, \ldots, \frac{u_{xxxx}}{u}\Big)\mathrm{d}x, \end{aligned}$$

where $S_0(\xi) = \xi_4/(\alpha-1)$. Clearly, performing integration by parts does not change the above integral value, but only the integrand function. Hence, if there exist a combination of coefficients c_1, c_2 and c_3 which makes the integrand function nonnegative, i.e. the polynomial $(S_0 + c_1T_1 + c_2T_2 + c_3T_3)(\xi)$ nonnegative for all $\xi \in \mathbb{R}^4$, then E_α is an entropy. The integral problem of proving an entropy dissipation has been translated into a decision problem about real polynomials. The big advantage of the latter is that it is allways solvable in an algorithmic way [13]. This is a famous result by Tarski, and the procedure is in real algebraic geometry known as quantifier elimination. For example, let $p(\xi_1,\xi_2)=a_1\xi_1^4+a_2\xi_1^2\xi_2+a_3\xi_2^2$ be a given polynomial. Then the quantified formula $\forall\xi_1,\xi_2\in\mathbb{R}\,:\,p(\xi_1,\xi_2)\geq 0$ is equivalent to the quantifier free statment: either $a_3>0$ AND $4a_1a_3-a_2^2\geq 0$ OR $a_3=a_2=0$ AND $a_1\geq 0$. The proof is elementary [11].

Our decision problem for finding entropies of the modified thin-film equation reads

$$(\exists\, c_1,c_2,c_3\in\mathbb{R})(\forall\xi\in\mathbb{R}^4)\;:\;(S_0+c_1T_1+c_2T_2+c_3T_3)(\xi)\geq 0. \tag{5}$$

In order to solve (5), we first simplify the polynomial. Notice, from the above homogeneity property of polynomials, that the highest power with respect to variables ξ_3 and ξ_4 is one, so that negative values could be attained. Therefore, those variables can be apriori eliminated by an appropriate choice of coefficients. Taking $c_3=-1/(\alpha-1)$ nulllifies variable ξ_4 in the polynomial, and further choice of $c_2=(\alpha+\beta-1)/(\alpha-1)$ nullifies the indefinite term $\xi_1\xi_3$. Thus, it remains to solve a simpler decision problem

$$(\exists\, c_1)(\forall\xi_1,\xi_2\in\mathbb{R})\;:\;(\alpha+\beta-3)c_1\xi_1^4+\Big(3c_1+\frac{(\alpha+\beta-2)(\alpha+\beta-1)}{\alpha-1}\Big)\xi_1^2\xi_2+\frac{\alpha+\beta-1}{\alpha-1}\xi_2^2\geq 0. \tag{6}$$

This problem can be solved directly using the above simple example and elementary algebra, or we can use the computer algebra system `Mathematica` and command `Reduce` therein, which implements the cylindrical algebraic decomposition (CAD) [7] to solve the quantifier elimination problem. Solving (6) results in an algebraic set of constraints on real parameters α and β, which reads: $3/2\leq\alpha+\beta\leq 3$ AND $\alpha>1$ OR $\alpha+\beta=1$ AND $\beta\neq 0$ (see Appendix). Hence, for given β, all α-functionals with α satisfing the above constraints, are entropies for the (MTF) equation.

The same procedure can be applied to obtain even stronger dissipation bounds,

$$-\frac{\mathrm{d}}{\mathrm{d}t}E_\alpha[u]\geq c\int_{\mathbb{T}}\Big(\big(u^{(\alpha+\beta)/4}\big)_x^4+\big(u^{(\alpha+\beta)/2}\big)_{xx}^2\Big)\,\mathrm{d}x, \tag{7}$$

for some strictly positive constant $c>0$. Observe that the right hand side in (7) also admits the polynomial representation $c\int_{\mathbb{T}}u^{\alpha+\beta}Q(u_x/u,u_{xx}/u)\mathrm{d}x$ with

$$Q(\xi_1,\xi_2)=\Big(\frac{\gamma^4}{256}+\frac{\gamma^2(\gamma-2)^2}{16}\Big)\xi_1^4+\frac{\gamma^2(\gamma-2)}{4}\xi_1^2\xi_2+\frac{\gamma^2}{4}\xi_2^2, \tag{8}$$

and $\gamma=\alpha+\beta$. Therefore, integral inequality (7) translates into the following decision problem

$$(\exists\, c_1\in\mathbb{R},c>0)(\forall\xi_1,\xi_2\in\mathbb{R})\;:S(\xi_1,\xi_2)-cQ(\xi_1,\xi_2)\geq 0, \tag{9}$$

where $S(\xi_1,\xi_2)$ is defined by the polynomial expression in (6). To solve the last decision problem, computer algebra system like `Mathematica` has to be invoked, and one finds that the α-functionals are dissipated according to (7) if: $3/2<\alpha+\beta<3$ AND $\alpha>1$.

We summarize our results about the modified thin-film equation into the following theorem.

Theorem 1 *Let u be smooth, strictly positive, 1-periodic solution to equation* (MTF)*. The functionals E_α are entropies provided that $3/2\leq\alpha+\beta\leq 3$ AND $\alpha>1$ OR $\alpha+\beta=1$ AND $\beta\neq 0$. Moreover, if $3/2<\alpha+\beta<3$ AND $\alpha>1$, entropy production inequality* (2) *holds for some $c>0$ with P_α defined by the right hand side in* (7)*.*

2.2 LUBRICATION EQUATIONS (SATF) AND (ATF)

Next we provide analogous results for other two lubrication equations: (SATF) and (ATF), that only differ with (MTF) in the polynomial S_0, which is characteristic for each equation. Calculating the time derivative of E_α with respect to solutions of (SATF) and (ATF), and applying integration by parts, we obtain

$$-\frac{\mathrm{d}}{\mathrm{d}t}E_\alpha[u]=\int_{\mathbb{T}}u^{\alpha+\beta}S_0\Big(\frac{u_x}{u},\ldots,\frac{u_{xxxx}}{u}\Big)\mathrm{d}x,$$

where $S_0(\xi)=-\xi_1\xi_3-\beta\xi_1^2\xi_2$ and $S_0(\xi)=-\xi_1\xi_3-3\beta\xi_1^2\xi_2-\beta(\beta-1)\xi_1^4$ for (SATF) and (ATF), respectively.

Solving decision problems like (5) and (9), we obtain the following results.

Theorem 2 *(i) Let u be smooth, strictly positive,* 1*-periodic solution to equation* (SATF)*, then the functionals E_α are entropies if $(\alpha+\beta-3)(\alpha-2\beta-3) \leq 0$. (ii) Let u be smooth, strictly positive,* 1*-periodic solution to equation* (ATF)*, then the functionals E_α are entropies provided that $(2\alpha-\beta-3)(\alpha-2\beta-3) \leq 0$.*

Furthermore, if the above conditions hold with strict inequalities, then in both cases, there exists a strictly positive constant $c > 0$ such that entropy production inequalities (2) *hold with P_α as in* (7)*.*

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APPENDIX

The following simple code in `Mathematica` performs quantifier elimination and solves the polynomial decision problem (6):

```
In[5]:= Reduce[
   Exists[c1,
    ForAll[{ξ1, ξ2}, (α + β - 3) c1 ξ1^4 + ((α + β - 2) (α + β - 1))/(α - 1) + 3 c1) ξ1^2 ξ2 + (α + β - 1)/(α - 1) ξ2^2 ≥
      0]], Reals] // FullSimplify

Out[5]= (α + β == 1 && β ≠ 0) || (β == 0 && 3/2 ≤ α ≤ 3) ||
  (3/2 ≤ α + β ≤ 3 && (0 < β < 1/2 || β < 0)) || (2 β ≥ 1 && α > 1 && α + β ≤ 3)
```

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